数理解析研究所講究録
745 巻 1991 年 202-219

Matched Asymptotic Expansion Method to Integral Formulations of Wing Theories.

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Summary

In various wing theories, governing relations are formed as an integral equation. They contain in general a small parameter such as thickness or camber of an aerofoil, inverse of aspect ratio of three dimensional wing, and jet momentum coefficient of a thin jet-flapped aerofoil, when we consider them as a perturbation theory. Then if we wish to solve them, we have some questions on the perturbation problem : how do we know whether this equation is singular or regular, and how do we obtain an asymptotic solution if it is singular. To answer these questions, we treat a first kind of the linear Fredholm integral equation whose kernel contains a small parameter and discuss the asymptotic behavior of its solution without knowledge of explicit representation of the solution. As a consequence of this study, provided that the integral operator does not have any significant local operator, this problem becomes regular. The necessary condition that this problem is singular is that the transformed operator of the degeneration of the integral operator to the significant local region is contained in the significant local operator. A method how to obtain an asymptotic solution on the singular case is proposed and its rationality is proved on the overlapped hypothesis.

1. Introduction.

The study of functions which are implicitly defined as solutions of a differential equation containing a small parameter and satisfying some supplementary conditions, such as boundary conditions or initial conditions, is carried out in detail in Eckhaus [1]. Further, in Van Dyke [2], Kevorkian and Cole [3], et al., a lot of problems confronted by engineers, physicists and applied mathematicians are treated and we can find a wealth of techniques and results in them.

Integral equations have been applied to various physical and engineering problems. In them, there are many cases : their kernel contains a small parameter, e.g., high aspect lifting surface problem and slender body problem [4,5], which are also treated as a perturbation problem on a differential equation [section 9.2 in Van Dyke [2], section 4.3.1 in Kevorkian and Cole [3]]. Comparing the above works [4,5] with [2,3], it may be seen that a perturbation approach of an integral equation has some merit, but the study of an integral equation whose kernel contains a small parameter on perturbation problems has not been carried out in detail, as long as the present author knows. In this paper, the first kind of a linear Fredholm integral equation which appears often in wing theories is treated :

$$\mathrm{K}_{\varepsilon} f \triangleq \int_0^1 K(x,y;\varepsilon) f(y;\varepsilon) dy = g(x;\varepsilon) \qquad (1)$$

where a given function $g(x;\varepsilon)$ is continuous in $x\in[0,1]$ and $\varepsilon\in(0,\varepsilon_0]$. If an asymptotic expansion of $f(x;\varepsilon)$ is obtained with some given small order sequences in any $x\in[0,1]$, then we call this problem " a regular perturbation problem ". If the above asymptotic expansion breaks down in some subdomain, we call it "a singular perturbation problem" and its domain is called "significant local region".

In the section 2 of this article, we describe the definitions of integral operators which we will use in this article, and we show main theorems. When Eq.(1) is regular, we show in Theorem 2.1 that the first approximation satisfies the following relation :

$$K_o f_o = g_o \quad \text{in } x\in[0,1] \qquad (2)$$

where K_o is the degeneration of the integral operator K_ε. When Eq.(1) is singular in the first approximation, $K_\varepsilon f_o$ does not exist for some $x\in[0,1]$. Then we have questions : will the first order regular approximation f_o be able to be governed by Eq.(2), and is it possible to know that this problem is singular without solving Eq.(2). For the latter question, we show in Theorem 2.2 the necessary condition that this problem is singular : The degeneration K_o^* of the significant local operator of K_ε on some local region contains the degeneration of the transformed operator of K_o to the same region. In the singular perturbation problem, there arises further question besides the former question : which equation is the significant local solution governed by. Theorem 2.4 of this paper shows a set of integral equations which must be governed by the regular and singular asymptotic solutions respectively, provided that the significant local region is near $x=0$.

$$\int_0^1 K_o(x,y)f_o(y)\mathrm{d}y = g_o(x) + \sum_{p=1} \hat{C}_p^{o}\hat{P}_p(\Lambda)x^{\lambda_p^+} \qquad (3)$$

$$\int_0^\infty K_o^*(X,\eta)f_o^*(\eta)d\eta = g_o^*(X) + \sum_{p=1} \hat{C}_p^{*o}\hat{Q}_p(\Lambda_X)X^{-\mu_p^-} \qquad (4)$$

where X is the significant local variable, integral signs $\oint_0^1[\ ]dy$ and $\oint_0^\infty[\ ]d\eta$ indicate the finite part of $\lim_{\Delta\to 0}\int_\Delta^1[\ ]dy$ and $\lim_{\Delta\to 0}\int_0^{1/\Delta}[\ ]d\eta$, respectively. Functions, $\hat{P}_p(\Lambda)$ and $\hat{Q}_p(\Lambda_X)$, are polynomials in Λ ($=\ln x$) and Λ_X ($=\ln X$), and $\hat{C}_p^o$, $\hat{C}_p^{*o}$ are constants which will be determined from the matching principle. In section 3 of this article, we prove the above theorems.

2. Definitions and main theorems.

We consider the first kind of linear Fredholm integral equation given by Eq.(1). We assume that a solution $f(x;\varepsilon)$ exists and $g(x;\varepsilon)$ is continuous in $x\in[0,1]$ and $\varepsilon\in(0,\varepsilon_o]$ where $\varepsilon_o(>0)$ is a small parameter. We further require that $g(x;\varepsilon)$ converges uniformly to $g_o(x)$ on $x\in[0,1]$ as $\varepsilon\to 0$, which is not identically zero. In the present paper, the uniform behavior of Definition 1.3.2 in Eckhaus [1] is taken as a measure of order of magnitude of functions. In this section, we suppose that the significant local region is near $x=0$ if it exists.

Definition of degeneration : The degeneration of K_ε is an integral operator K_o, not identically zero, such that for all test function $\theta(x)\in C_o^\infty$ on $x\in[0,1]$, independent of ε, for which $K_\varepsilon\theta$ exists and is not identically zero and for some order function $\delta_o(\varepsilon)$, we have ; $\lim_{\varepsilon\to 0} K_\varepsilon\theta/\delta_o = K_o\theta$.

Definitions of transformed operator and local operator : Let define a continuous one to one transformation from x to the local variable X, $x=\phi X$, where $\phi\to 0$ as $\varepsilon\to 0$. The transformed operator, T_X, is defined as changing the variable x to X ; $\mathrm{T}_X f = f(\phi X;\varepsilon)$. The local operator, K_ε^*, is defined by

$$\mathrm{K}_\varepsilon^*\theta^* = \int_0^{1/\phi} \phi \mathrm{T}_X \mathrm{T}_\eta K \circ \theta^* d\eta$$

where η is also the local integral variable , $\eta = y/\phi$, and θ^* is a test function of X in $X \in [0,\infty)$.

<u>Definition of operator **F**</u> : Let N be a large positive number independent of ε. The operator **F** is defined as

$$\mathrm{F}\mathrm{K}_\varepsilon^*\theta^* = \int_0^N K_\varepsilon^*\theta^* d\eta$$

where K_ε^* is a kernel of K_ε^* ; $K_\varepsilon^* = \phi \mathrm{T}_X \mathrm{T}_\eta K$.

<u>Definition of contained operator</u> : Let $\mathrm{K}_\mathrm{o}^{*(1)}$ and $\mathrm{K}_\mathrm{o}^{*(2)}$ be the degenerations of the local operator of K_ε to the same local region, which are respectively defined by the continuous mapping ; $x = \phi_1 X_1$ and $x = \phi_2 X_2$, where $\phi_1 \neq \phi_2$ and $\phi_1, \phi_2 \to 0$ as $\varepsilon \to 0$. Let $\mathrm{K}_\mathrm{o}^{*(1,2)}$ be $\mathrm{T}_{X_2}\mathrm{K}_\mathrm{o}^{*(1)}$. We shall say that $\mathrm{K}_\mathrm{o}^{*(2)}$ is contained in $\mathrm{K}_\mathrm{o}^{*(1)}$ if for all function θ^* independent of ε for which $\mathrm{F}\mathrm{K}_\mathrm{o}^{*(1,2)}\theta^*$ exists and is not identically zero and for some order function δ, one has $\lim_{\varepsilon\to 0} \mathrm{F}\mathrm{K}_\mathrm{o}^{*(1,2)}\theta^*/\delta = \mathrm{F}\mathrm{K}_\mathrm{o}^{*(2)}\theta^*$.

<u>Definition of significant local operator</u> : Suppose that there exists degeneration of the local operator K_ε^*. Let us define a set of ϕ, say S, of the continuous one to one mapping ; $x = \phi X$, where $\phi \to 0$ as $\varepsilon \to 0$. Let $\mathrm{K}_\mathrm{o}^*(\phi)$ be the degeneration of the local operator due to ϕ. If there exists some mapping $\phi_\mathrm{o} \in \mathrm{S}$ such that $\mathrm{K}_\mathrm{o}^*(\phi_\mathrm{o})$ would not be contained by any other $\mathrm{K}_\mathrm{o}^*(\phi)$ $(\phi \in \mathrm{S})$, $\mathrm{K}_\mathrm{o}^*(\phi_\mathrm{o})$ is called

the significant local operator.

In the present paper, we suppose that the integral operator K_ε of Eq.(1) is degenerated by $K_\varepsilon = \delta_o K_o + K_p$ where δ_o is some order function of ε. Then we have for any test function $\theta \in C_o^\infty$ in $x \in [0,1]$, which is independent of ε : $\lim_{\varepsilon\to 0} K_p\theta/\delta_o = 0$. If we further assume that the adjoint integral operator $\hat{K}_p$ of K_p exists, we have : $\lim_{\varepsilon\to 0} \hat{K}_p\theta/\delta_o = 0$. We state our main theorems in this section and their proofs will be stated in the next section. The following Theorem 2.1 says that the first approximation is governed by $K_o f_o = g_o$ in a regular perturbation problem.

Theorem 2.1. Suppose that there exists a function $f_o(x)$ such that $\lim_{\varepsilon\to 0} \delta_o f(x;\varepsilon) = f_o(x)$ uniformly in $[0,1]$ and $K_o f_o$ exists as a continuous function. Then we have $K_o f_o = g_o$ in $[0,1]$.

We suppose that the region near $x=0$ is only significant if the significant local region exists and that the local operator K_ε^* which is given by the local variable $X = x/\phi$ where $\phi\to 0$ for $\varepsilon\to 0$ is degenerated by $K_\varepsilon^* = \delta_o^* K_o^* + K_p^*$, $\lim_{\varepsilon\to 0} F K_p^* \theta^*/\delta_o^* = 0$, where δ_o^* is some order function of ε, and θ^* is a test function of X. We define the local functions $f^*(X;\varepsilon)$ and $g^*(X;\varepsilon)$ by $f^*(X;\varepsilon) = T_X f$, $g^*(X;\varepsilon) = T_X g$. Then the following theorem shows the necessary condition that $f_o(x)$ does not converge to $f_o^*(X)$ uniformly with fixed $X \in [0,\infty)$.

Theorem 2.2. We assume : (1) $f^*(X;\varepsilon)\to f_o^*(X)$ and $g^*(X;\varepsilon)\to \delta_o^* g_o^*(X)$ uniformly with fixed X $(\in[0,1/\hat{d}])$ for $\varepsilon\to 0$, where $\hat{d}$ is an arbitrary ε-independent small parameter with $0<\hat{d}<1$. (2) $K_o f_o$ and $g_o(x)$ are continuous in $x\in[d,1]$, where d is an arbitrary ε-independent small parameter with $0<d<1$. (3) $K_o^* f_o^*$ and $g_o^*(X)$ are also continuous in $X\in[0,1/\hat{d}]$. (4) We have for $\varepsilon\to 0$,

$$\int_{1/\psi}^{1/\phi} [(T_X K_o)_o(X,\eta) - K_o^*(X,\eta)] T_\eta f_o d\eta = o(\delta_o^*)$$

where ψ is a small parameter with $\psi\to 0$ and $\phi/\psi\to 0$ for $\varepsilon\to 0$, and $(T_X K_o)_o(X,\eta)$ is the kernel of the integral operator $(T_X K_o)_o$, which is the degeneration of $T_X K_o$ (i.e., $T_X K_o = \delta_o^*(T_X K_o)_o + (T_X K_o)_p$). The necessary condition that $f_o(x)$ does not converge to $f_o^*(X)$ uniformly with fixed X ($\in[0,\infty)$) for $\varepsilon\to 0$ is that $(T_X K_o)_o$ is contained in K_o^*.

This theorem is extended easily to arbitrary domain D independent on ε. Let us consider that K_ε has a significant local operator K_ε^* to some local region D_o^* for some local variable X. Then the degeneration of $T_X K_o$ must be contained in the significant local operator K_o^*. From this theorem, we may see both the significant local region and the significant local transformation to this region, i.e., the significant local variable. The proof of this theorem is based on the following Extension Lemma whose proof will be stated in the next section.

Lemma 2.1. Let an integral operator K_ε be defined by

$$K_\varepsilon\theta = \int_d^1 K(x,y;\varepsilon)\,\theta(y)dy, \qquad K_\varepsilon = \delta_o K_o + K_p$$

where $\varepsilon\in(0,\varepsilon_o]$, θ is a test function on $[0,1]$, and d is an ε-independent parameter with $0<d<1$. Assume that $K_\varepsilon\theta$ is continuous on $[0,1]\times(0,\varepsilon_0]$ for any d. Then there exists an order function $\delta(\varepsilon)=o(1)$ such that

$$\int_\delta^1 K(x,y;\varepsilon)\,\theta(y)dy = \delta_0 \int_\delta^1 K_0(x,y)\theta(y)dy + o(\delta_0) \quad \text{in } [\delta,1]$$

where function K_0 is the kernel of K_0.

Let an integral operator K_ε^* be defined by

$$\mathrm{K}_\varepsilon^*\theta^* = \int_0^{1/\psi} K_\varepsilon^*(X,\eta;\varepsilon)\theta^*(\eta)d\eta$$

where $\psi(\varepsilon)$ is a positive monotonic function such that $\psi(\varepsilon)\to 0$ as $\varepsilon\to 0$. Assume that $\mathrm{FK}_\varepsilon^*\theta^*$ is continuous on $(X,\varepsilon)\in[0,N]\times(0,\varepsilon_0]$ for any N which is large, positive, and ε-independent. Let FK_0^* be the degeneration of $\mathrm{FK}_\varepsilon^*$: $\mathrm{FK}_\varepsilon^* = \delta_0^*\mathrm{FK}_0^* + \mathrm{FK}_p^*$. Then there exists an order function $\hat{\delta}(\varepsilon)=o(1)$ such that

$$\int_0^{1/\hat{\delta}} K_\varepsilon^*(X,\eta;\varepsilon)\theta^*(\eta)d\eta = \delta_0^* \int_0^{1/\hat{\delta}} K_0^*(X,\eta)\theta^*(\eta)d\eta + o(\delta_0^*) \quad \text{in } [0,1/\hat{\delta}]$$

where K_0^* is the kernel of K_0^*.

In the singular case, we have to consider which relations should govern the regular and significant local solutions. We here assume that the significant local region is near $x=0$. Then we have :

Theorem 2.3. We assume : (1) The same condition as of Theorem 2.2 is satisfied. (2) The kernel is expressed by

$$K(x,y;\varepsilon) = \delta_0[\sum_{p=1} A_p(y;\varepsilon)P_p(\Lambda)x^{-\lambda_p^-} + \sum_{p=0} \hat{A}_p(y;\varepsilon)\hat{P}_p(\Lambda)x^{\lambda_p^+}] \quad \text{for } \phi(\varepsilon)\ll y\ll x\ll 1$$

$$K(x,y;\varepsilon) = \delta_0[\sum_{p=0} B_p(y;\varepsilon)Q_p(\Lambda)x^{\mu_p^+} + \sum_{p=1} \hat{B}_p(y;\varepsilon)\hat{Q}_p(\Lambda)x^{-\mu_p^-}] \quad \text{for } \phi(\varepsilon)\ll x\ll y\ll 1 \tag{5}$$

where $0<\lambda_p^{\pm}<\lambda_{p'}^{\pm}$, and $0<\mu_p^{\pm}<\mu_{p'}^{\pm}$, for $p<p'$, $P_p(\Lambda)$, $\hat{P}_p(\Lambda)$, $Q_p(\Lambda)$, and $\hat{Q}_p(\Lambda)$ are polynomials in Λ $(=\ln x)$, and the set of p such that A_p , $\hat{A}_p$, B_p, and $\hat{B}_p$ for $\varepsilon\to 0$ are not identically zero is infinite. (3) Operators K_ε and K_ε^* are expressed by

$$K_\varepsilon = \delta_0 K_0 + \delta_1 K_1 + \ldots, \qquad K_\varepsilon^* = \delta_0^* K_0^* + \delta_1^* K_1^* + \ldots$$

where $\delta_i(\varepsilon)$ and $\delta_i^*(\varepsilon)$ $(i=0,1,..)$ are order functions. (4) The given functions, $g(x;\varepsilon)$ and $g^*(X;\varepsilon)$, are expressed by

$$g(x;\varepsilon) = \hat{\delta}_0 g_0(x) + \hat{\delta}_1 g_1(x) + \ldots, \quad g^*(X;\varepsilon) = \hat{\delta}_0^* g_0^*(X) + \hat{\delta}_1^* g_1^*(X) + \ldots$$

where $\hat{\delta}_i(\varepsilon)$ and $\hat{\delta}_i^*(\varepsilon)$ $(i=0,1,..)$ are order functions, which are satisfied by $\hat{\delta}_{m-i}\delta_i=\hat{\delta}_m\delta_0$ and $\hat{\delta}_{m-i}^*\delta_i^*=\hat{\delta}_m^*\delta_0^*$ $(i=0,1,..,m;m=0,1,..)$. Then f, f^* are given by

$$f(x;\varepsilon) = [\hat{\delta}_0 f_0(x) + \hat{\delta}_1 f_1(x) + \ldots]/\delta_0$$

$$f^*(X;\varepsilon) = [\hat{\delta}_0^* f_0^*(X) + \hat{\delta}_1^* f_1^*(X) + \ldots]/\delta_0^*$$

and there exist integers s_i $(i=1,2,3,4)$ such that for a pre-assigned order $\delta^{(r)}$ and $\delta^{*(r)}$;

$$\int_0^1 K_0(x,y)f_m(y)dy = g_m(x) - \sum_{p=1}^{m}\int_0^1 K_p(x,y)f_{m-p}(y)dy + \sum_{p=0}^{s_1} C_p^m P_p(\Lambda)x^{-\lambda_p^-}$$
$$+ \sum_{p=1}^{s_2}\hat{C}_p^m \hat{P}_p(\Lambda)x^{\lambda_p^+} + o(\delta^{(r)}) \qquad (6)$$

$$\int_0^\infty K^*(X,\eta)f_m^*(\eta)d\eta = g_m^*(X) - \sum_{p=1}^{m}\int_0^\infty K_p^*(X,\eta)f_{m-p}^*(\eta)d\eta + \sum_{p=0}^{s_3} C_p^{*m} Q_p(\Lambda_X)X^{\mu_p^+}$$
$$+ \sum_{p=1}^{s_4}\hat{C}_p^{*m}\,\hat{Q}_p(\Lambda_X)X^{-\mu_p^-} + o(\delta^{*(r)}) \qquad (7)$$

where C_p^m, $\hat{C}_p^m$, C_p^{*m} and $\hat{C}_p^{*m}$ are constants, $\Lambda_X(=\ln X)$.

<u>Theorem 2.4.</u> In the same assumptions as in Theorem 2.3, some regular and significant local expansions of $f(x;\varepsilon)$ are the solutions of Eqs.(6) and (7) respectively, if they exist.

The constants, C_p^m, C_p^{*m}, $\hat{C}_p^m$ and $\hat{C}_p^{*m}$, are determined if $f_m(x)$ and $f_m^*(X)$ are obtained. We first assume these coefficients are unknown and second we obtain $f_m(x)$, $f_m^*(X)$ from Eqs.(6) and (7). Finally we determine them by using the matching principle [cf. Van Dyke [2]]. Thus, the asymptotic expansions are obtained. This approach is the same proposed by the present author with his coworker [4,6,7].

Theorem 2.5. We suppose : (1) The overlap hypothesis and the same conditions as in Theorem 2.3 are satisfied. (2) The given function, $g(x;\varepsilon)$, is expressible as a composite form. (3) The regular and significant local solutions of Eqs.(6) and (7) are obtained under the assumption that C_p^m, $\hat{C}_p^m$, C_p^{*m}, and $\hat{C}_p^{*m}$ are unknown, and (4) these coefficients are determined from the matching condition. Then their composite form is one of an asymptotic solution of Eq.(1).

3. Proofs of Theorems and Lemma 2.1.

In this section, we show the proofs of Theorem 2.1-2.4 and Lemma 2.1.

Proof of Theorem 2.1. We consider the inner product for a test function θ. Then Eq.(1) is identically expressed by $(K_\varepsilon f,\theta)=\delta_o(K_o f,\theta)+(K_p f,\theta)$. Introducing the adjoint operators $\tilde{K}_o$ and $\tilde{K}_p$ of K_o and K_p respectively, we have : $(K_\varepsilon f,\theta)=\delta_o(f,\tilde{K}_o\theta)+(f,\tilde{K}_p\theta)$. Since $\tilde{K}_p\theta/\delta_o\to 0$ and $f(x;\varepsilon)\to f_o(x)/\delta_o$ uniformly as $\varepsilon\to 0$, we have

$$(K_\varepsilon f,\theta)\to(f_o,\tilde{K}_o\theta)=(K_o f_o,\theta)\ \text{as}\ \varepsilon\to 0$$

We have also ; $(g,\theta)\to(g_o,\theta)$ as $\varepsilon\to 0$, because $g(x;\varepsilon)\to g_o(x)$

uniformly as $\varepsilon\to 0$. Further, $g_o(x)$ and $K_o f_o$ are continuous from the condition of theorem, so we conclude : $K_o f_o = g_o$ in $[0,1]$.

Proof of Theorem 2.2. We assume at first that $f_o(x)$ converges to $f_o^*(X)$ uniformly with fixed any $X\in[0,\infty)$ for $\varepsilon\to 0$, and we show finally to arrive at the contradiction. We introduce the inner product for a test function $\theta^*\in C_o^\infty$ in $[0,\infty)$ such that $\tilde{T_X K_o}\theta^*$ and $\tilde{K}_o^*\theta^*$ exist where $\tilde{T_X K_o}$ and $\tilde{K}_o^*$ are adjoint of $T_X K_o$ and K_o^* respectively. Then we have in $[0,N]$ where N is arbitrary, large, positive, and ε-independent ;

$$\int_0^N \theta^* T_X g_o dX = \int_0^N \theta^* dX \int_0^1 T_X K_o f_o dy = \int_0^N \theta^* dX \int_0^{1/\phi} \phi\ T_\eta T_X K_o \cdot T_\eta f_o d\eta \to$$

$$\delta_o^* \int_0^{1/\phi} T_\eta f_o d\eta \int_0^N (\tilde{T_X K_o})_o \theta^* dX = \delta_o^* \int_0^N \theta^* dX \int_0^{1/\phi} (T_X K_o)_o T_\eta f_o d\eta$$

From this result and Lemma 2.1, there exists a small parameter ψ ($\psi\to 0$ as $\varepsilon\to 0$) such that

$$\int_0^{1/\psi} \theta^* dX \int_0^1 T_X K_o f_o dy \to \delta_o^* \int_0^{1/\psi} \theta^* dX \int_0^{1/\phi} (T_X K_o)_o T_\eta f_o d\eta$$

On the other hand, since $K_o^* f_o^* = g_o^*$, we have

$$\int_0^{1/\psi} \theta^* g_o^* dX = \int_0^{1/\psi} \theta^* dX \int_0^\infty K_o^* f_o^* d\eta = \int_0^{1/\psi} \theta^* dX \int_0^{1/\psi} K_o^* f_o^* d\eta$$

$$+ \int_0^{1/\psi} \theta^* dX \int_{1/\psi}^{1/\phi} K_o^* f_o^* d\eta + \int_0^{1/\psi} \theta^* dX \int_{1/\phi}^\infty K_o^* f_o^* d\eta \to \int_0^{1/\psi} \theta^* dX \int_0^{1/\psi} K_o^* f_o^* d\eta$$

$$+ \int_0^{1/\psi} \theta^* dX \int_{1/\psi}^{1/\phi} K_o^* f_o^* d\eta \qquad \text{as } \varepsilon \to 0$$

Since $T_X g_o \to \delta_o^* g_o^*$ and $T_X f_o \to f_o^*$ uniformly as $\varepsilon\to 0$, we must satisfy the following relation :

$$\int_0^{1/\psi} \theta^* dX \int_0^{1/\psi} (T_X K_o)_o T_\eta f_o d\eta \to \int_0^{1/\psi} \theta^* dX \int_0^{1/\psi} K_o^* T_\eta f_o d\eta$$

$$+\int_0^{1/\psi}\theta^* dX\int_{1/\psi}^{1/\phi}[K_0^* - (T_X K_0)_0]T_\eta f_0 d\eta \qquad \text{as } \varepsilon \to 0$$

If $\int_N^\infty (T_X K_0)_0 T_\eta f_0 d\eta$ exists for any large positive N, we have from the above relation and condition (4) of theorem

$$\int_0^{1/\psi}\theta^* dX\int_0^{1/\psi}[(T_X K_0)_0 - K_0^*]T_\eta f_0 d\eta \to 0 \quad \text{as } \varepsilon \to 0$$

Therefore, we arrive at $T_X^{\sim} K_0\theta^* \to K_0^*\theta^*$ as $\varepsilon\to 0$, provided that the following relation does not satisfy ;

$$\int_0^{1/\psi}[(T_X K_0)_0 - K_0^*]T_\eta f_0 d\eta \to 0 \qquad (*)$$

This means that $T_X K_0\theta^* \to K_0^*\theta^*$ as $\varepsilon\to 0$, so that K_0^* is contained in $T_X K_0$. This is a contradiction because K_0^* is significant. If $\int_N^\infty (T_X K_0)_0 T_\eta f_0 d\eta$ does not exist, $T_X g_0$ does not converge $\delta_0^* g_0^*$ uniformly, so that this is contradiction. If the relation (*) is satisfied, then we can not say alway that this problem becomes singular.

<u>Proof of Lemma 2.1.</u> We define $\Phi(x;\varepsilon;d)$ and $\Phi_0(x;d)$ by

$$\Phi(x;\varepsilon;d) = \frac{1}{\delta_0}\int_d^1 K(x,y;\varepsilon)\theta(y)dy\,, \qquad \Phi_0(x;d) = \int_d^1 K_0(x,y)\theta(y)dy$$

where θ is a continuous test function such that Φ and Φ_0 exist and are continuous in $x\in[d,1]$. Then we have from the definition : $\lim_{\varepsilon\to 0}|\Phi-\Phi_0| = 0$. We define $\tilde{g}(\varepsilon;d)$ for $0<d<1$ by

$$\tilde{g}(\varepsilon;d) = \sup_{x\in[d,1]}|\Phi-\Phi_0|$$

Then $\tilde{g}(\varepsilon;d)\leq\tilde{g}(\varepsilon;d')$ for $d'<d$ and $\tilde{g}(\varepsilon;d)\to 0$ as $\varepsilon\to 0$. Therefore, the first part of this lemma is proved from Lemma 2.2.1 in Eckhaus [1]. For the second part, we define $\Phi(X;\varepsilon;d)$ and $\Phi_0(X;d)$, and $\tilde{g}(\varepsilon;d)$ for $0<d<1$

$$\Phi(X;\varepsilon;d) = \frac{1}{\delta_o^*}\int_0^{1/d} K_\varepsilon^*(X;\eta;\varepsilon)\theta^*(\eta)d\eta$$

$$\Phi_o(X;d) = \int_0^{1/d} K_o^*(X;\eta)\theta^*(\eta)d\eta$$

$$\tilde{g}(\varepsilon;d) = \sup_{X\in[0,1/d]} |\Phi-\Phi_o|$$

Then we can also prove the second part of the present theory by using Lemma 2.2.1 in Eckhaus [1].

__Proof of Theorem 2.3.__ Taking into account that $K(x,y;\varepsilon)$ satisfies the expansion form of this theorem, we may see from Lemma 2.1 that there exists a parameter δ such that for $x>\delta$ with $\delta \succ \phi$ we have :

$$\int_0^\delta K(x,y;\varepsilon)f(y;\varepsilon)dy = \delta_o[\sum_{p=1} P_p(\Lambda)x^{-\lambda_p^-}\int_0^\delta A_p(y;\varepsilon)f(y;\varepsilon)dy$$

$$+\sum_{p=0}\hat{P}_p(\Lambda)x^{\lambda_p^+}\int_0^\delta \hat{A}_p(y;\varepsilon)f(y;\varepsilon)dy] = \delta_o[\sum_{p=1}P_p(\Lambda)x^{-\lambda_p^-}C_p(\varepsilon)$$

$$+\sum_{p=0}\hat{P}_p(\Lambda)x^{\lambda_p^+}\hat{C}_p(\varepsilon)]$$

where notation $\succ$ is the Hardy's notation which is defined in Eckhaus [1], and

$$C_p(\varepsilon) = \int_0^\delta A_p(y;\varepsilon)f(y;\varepsilon)dy, \qquad \hat{C}_p(\varepsilon) = \int_0^\delta \hat{A}_p(y;\varepsilon)f(y;\varepsilon)dy$$

From the definition of K_n, the kernel, $K_n(x,y)$, of K_n is also expressible as the same expansion form as $K(x,y;\varepsilon)$. Therefore, we have

$$\sum_{p'=0}^{m}\int_0^\delta K_{p'}(x,y)f_{m-p'}(y)dy = \delta_o[\sum_{p=1} P_p(\Lambda)x^{-\lambda_p^-}\tilde{C}_p(\varepsilon)$$

$$+\sum_{p=0}\hat{P}_p(\Lambda)x^{\lambda_p^+}\tilde{C}_p(\varepsilon)]$$

where $\tilde{C}_p(\varepsilon)$ and $\hat{C}_p(\varepsilon)$ are determined if $f_n(x)$ is given. From substituting these relations and asymptotic expressions of $f(x)$ and $g(x)$ into Eq.(1), we may have Eq.(6) for a pre-assigned order $\delta^{(r)}$. From the similar steps we may have Eq.(7).

Proof of Theorem 2.4. If a regular expansion and a significant local expansion exist, it may be seen from the derivation of Eqs.(6) and (7) in the proof of Theorem 2.3 that they satisfy these equations, respectively.

Proof of Theorem 2.5. Let us define operator $E_x^{(m)}$ as does in Eckhaus [1]. Then there exists an integer m such that for $\delta_m^{(r)}$ which is any element of a pre-assigned ordered sequence of order functions ; for $x\in[d,1]$,

$$f(x;\varepsilon) = E_x^{(m)}f + o(\delta_m^{(r)}), \qquad g(x;\varepsilon) = E_x^{(m)}g + o(\delta_m^{(r)})$$

where d is an arbitrary small ε-independent parameter with $0<d<1$. From Extension Theory 2.2.3 given in Eckhaus [1], there exists some order function $\delta_p(\varepsilon)$ ($\delta_p\to 0$ as $\varepsilon\to 0$) for given $\delta_m^{(r)}$ such that $f(x;\varepsilon)= E_x^{(m)}f + o(\delta_m^{(r)})$, $g(x;\varepsilon)=E_x^{(m)}g + o(\delta_m^{(r)})$ in $[\delta_p,1]$. From Lemma 2.1, there exists some order function $\delta_q(\varepsilon)$ ($\delta_q\to 0$ as $\varepsilon\to 0$) and we can choose integers m, s_1, and s_2 for any element of $\delta_k^{(r)}$, such that from Eq.(6)

$$\int_{\delta_q}^{1} K(x,y;\varepsilon)E_y^{(m)}f\,dy = E_x^{(m)}g + \delta_0\Big[\sum_{p=1}^{s_1} C_p^m P_p(\Lambda)x^{-\lambda_p^-}$$

$$+ \sum_{p=0}^{s_2} \hat{C}_p^m \hat{P}_p(\Lambda)x^{\lambda_p^+}\Big] + o(\delta_k^{(r)}) \quad \text{in } [\delta_p,1]$$

We define $\delta(\varepsilon)$ ($\delta\to 0$ as $\varepsilon\to 0$) by a lower order function between δ_q and δ_p. Then we have

$$\int_\delta^1 K(x,y;\varepsilon)E_y^{(m)}f\,dy = E_x^{(m)}g + \delta_o[\sum_{p=1}^{s_1} C_p^m P_p(\Lambda)x^{-\lambda_p^-}$$
$$+ \sum_{p=0}^{s_2} \hat{C}_p^m \hat{P}_p(\Lambda)x^{\lambda_p^+}] + o(\delta_k^{(r)}) \quad \text{in } [\delta,1] \qquad (8)$$

Following similar steps on Eq.(7), there also exist integers, m, s_3, and s_4 for $\delta_k^{(r)}$ such that

$$\int_0^{1/\tilde{\delta}} K^*(X,\eta;\varepsilon)E_\eta^{(m)}f^*d\eta = E^{(m)}g^* + \delta_o^*[\sum_{p=0}^{s_3} C_p^{*m}Q_p^*(\Lambda_X)X^{\mu_p^+}$$
$$+ \sum_{p=1}^{s_4} \tilde{C}_p^{*m}\hat{Q}_p^*(\Lambda_X)X^{-\mu_p^-}] + o(\delta_k^{(r)}) \quad \text{in } [0,1/\tilde{\delta}] \qquad (9)$$

where $\tilde{\delta}$ is an order function of ε with $\tilde{\delta}\to 0$ as $\varepsilon\to 0$. On the other hand, we easily see that there exist functions $F(x;\varepsilon)$ and $\hat{F}(X;\varepsilon)$ such that

$$\delta_o[\sum_{p=1}^{s_1} C_p^m P_p(\Lambda)x^{-\lambda_p^-} + \sum_{p=0}^{s_2} \hat{C}_p^m \hat{P}_p(\Lambda)x^{\lambda_p^+}] = \int_0^\delta K(x,y;\varepsilon)F(y;\varepsilon)dy$$
$$+ o(\delta_k^{(r)}) \qquad \text{in } [\delta,1] \qquad (10)$$

$$\delta_o^*[\sum_{p=0}^{s_3} C_p^{*m}Q_p^*(\Lambda_X)X^{\mu_p^+} + \sum_{p=1}^{s_4} \hat{C}_p^{*m}\hat{Q}_p^*(\Lambda_X)X^{-\mu_p^-}] = \int_{1/\tilde{\delta}}^{1/\phi} K_\varepsilon^*(X,\eta;\varepsilon)\hat{F}(\eta;\varepsilon)d\eta$$
$$+ o(\delta_k^{(r)}) \quad \text{in } [0,1/\tilde{\delta}] \qquad (11)$$

We note that these functions F and $\hat{F}$ are unknown because C_p^m, $\hat{C}_p^m$, C_p^{*m}, and $\hat{C}_p^{*m}$ are unknown. From Eqs.(9) and (11), we have

$$\int_0^{\phi/\tilde{\delta}} K(x,y;\varepsilon)T_yE_Y^{(m)}f^*dy = T_XE_X^{(m)}g^* + \int_{\phi/\tilde{\delta}}^1 K(x,y;\varepsilon)T_y\hat{F}dy$$
$$+ o(\delta_k^{(r)}) \qquad \text{in } [0,\phi/\tilde{\delta}] \qquad (12)$$

Noting that from assumption (1) of theorem, $\phi/\tilde{\delta}\succ\delta$, we consider the left hand side of Eq.(8) :

$$\int_\delta^1 K(x,y;\varepsilon)E_y^{(m)}f\,dy = (\int_\delta^{\phi/\tilde{\delta}} + \int_{\phi/\tilde{\delta}}^1)K(x,y;\varepsilon)E_y^{(m)}f\,dy = \int_\delta^{\phi/\tilde{\delta}} K(x,y;\varepsilon)E^{(m)}f\,dy$$

$$+ \delta_o[\sum_{p=0} x^{\mu_p^+} Q_p(\Lambda)\int_{\phi/\tilde{\delta}}^1 B_p(y;\varepsilon)E_y^{(m)} f dy + \sum_{p=1} x^{-\mu_p^-}\hat{Q}_p(\Lambda)\int_{\phi/\tilde{\delta}}^1 \hat{B}_p(y;\varepsilon)E_y^{(m)} f dy]$$

in $[\delta,\phi/\tilde{\delta}]$

The right hand side of Eq.(8) is related by Eq.(10). Therefore, we have on $x\in[\delta,\phi/\tilde{\delta}]$

$$\int_\delta^{\phi/\tilde{\delta}} K(x,y;\varepsilon)E_y^{(m)} f dy = E_X^{(m)} g + \delta_o[\sum_{p=1} P_p(\Lambda)x^{-\lambda_p^-}\int_0^\delta A_p(y;\varepsilon)F(y;\varepsilon)dy$$

$$+ \sum_{p=0} \hat{P}_p(\Lambda)x^{\lambda_p^+}\int_0^\delta \hat{A}_p(y;\varepsilon)F(y;\varepsilon)dy] - \delta_o[\sum_{p=0} Q_p(\Lambda)x^{\mu_p^+}\int_{\phi/\tilde{\delta}}^1 B_p(y;\varepsilon)E^{(m)} f dy$$

$$+ \sum_{p=1} \hat{Q}_p(\Lambda)x^{-\mu_p^-}\int_{\phi/\tilde{\delta}}^1 \hat{B}_p(y;\varepsilon)E_y^{(m)} f dy] + o(\delta_k^{(r)}) \qquad (13)$$

Following similar steps, we have by using Eq.(12) :

$$\int_\delta^{\phi/\tilde{\delta}} K(x,y;\varepsilon)T_y E_Y^{(m)} f^* dy = T_X E_X^{(m)} g^* + \delta_o[\sum_{p=0} Q_p(\Lambda)x^{\mu_p^+}\int_{\phi/\tilde{\delta}}^1 B_p(y;\varepsilon)T_y \hat{F} dy$$

$$+ \sum_{p=1} \hat{Q}_p(\Lambda)x^{-\mu_p^-}\int_{\phi/\tilde{\delta}}^1 \hat{B}_p(y;\varepsilon)T_y \hat{F} dy] - \delta_o[\sum_{p=1} P_p(\Lambda)x^{-\lambda_p^-}\int_0^\delta A_p(y;\varepsilon)T_y E_Y^{(m)} f^* dy$$

$$+ \sum_{p=0} \hat{P}_p(\Lambda)x^{\lambda_p^+}\int_0^\delta \hat{A}_p(y;\varepsilon)T_y E^{(m)} f^* dy] + o(\delta_k^{(r)}) \qquad (14)$$

Since the matching principle is satisfied, there further exists an integer m for a pre-assigned order $\delta_k^{(r)}$ such that

$$\int_\delta^{\phi/\tilde{\delta}} K(x,y;\varepsilon)[E_y^{(m)} f - T_y E_Y^{(m)} f^*]dy = o(\delta_k^{(r)}) \qquad (15)$$

Let us consider the case where $\lambda_p^\pm = \mu_p^\pm$. If we take $\tilde{m}=\max(m,\hat{m})$ and we define $\tilde{m}$ as m again, there exist some integers s_i (i=1,2,3,4) from Eqs.(13), (14), and (15) ;

$$\sum_{p=1}^{s_1} P_p(\Lambda)x^{-\lambda_p^-}\int_0^\delta A_p(y;\varepsilon)[F(y;\varepsilon) + T_y E_Y^{(m)} f^*]dy - \sum_{p=1}^{s_4} \hat{Q}_p(\Lambda)x^{-\lambda_p^-}$$

$$\times\int_{\phi/\tilde{\delta}}^1 \hat{B}_p(y;\varepsilon)[E_y^{(m)} f + T_y \hat{F}]dy = o(\delta_k^{(r)})$$

$$\sum_{p=0}^{s_3} Q_p(\Lambda)x^{\lambda_p^+}\int_{\phi/\tilde{\delta}}^{1} B_p(y;\varepsilon)[T_y\hat{F} + E_y^{(m)}f]dy - \sum_{p=0}^{s_2} \hat{P}_p(\Lambda)x^{\lambda_p^+}$$

$$\times\int_0^{\delta}\hat{A}_p(y;\varepsilon)[F(y;\varepsilon) + T_yE_Y^{(m)}f^*]dy = o(\delta_k^{(r)})$$

Therefore, we have :

$$F(x;\varepsilon) = -T_xE_X^{(m)}f^* + o(\delta_k^{(r)}) \qquad T_x\hat{F} = -E_x^{(m)}f + o(\delta_k^{(r)})$$

Because there is a set of p such that $P_p \neq \hat{Q}_p$, $A_p \neq \hat{B}_p$, $\hat{P}_p \neq Q_p$, or $\hat{A}_p \neq B_p$, since Eq.(1) is assumed to be singular. If $\lambda_p^+ \neq \mu_p^+$ or $\lambda_p^- \neq \mu_p^-$, then we arrive at the above relations also. Therefore, we have

$$\int_0^1 K(x,y;\varepsilon)[E_y^{(m)}f + T_yE_X^{(m)}f^* - E_y^{(m)}E_X^{(m)}f^*]dy = E_x^{(m)}g + T_xE_X^{(m)}g^*$$

$$- E_x^{(m)}E_X^{(m)}g^* + o(\delta^{(r)}) \text{ in } [0,1]$$

Acknowledgment : The author expresses his sincere thanks to Professor Eckhaus and Professor Van Dyke for discussing the earlier versions of the present works. He gratefully acknowledges the help that he received from Professor Fife in this work and Professor Yearout for his advise on the English. He also expresses his sincere thanks to Dr. S. Taniguti (Kyusyu University) for advising on the structure of this version.

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